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LETTER TO THE EDITOR

Equilibrium states of the spin glass on a Bethe lattice

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Abstract. The existence of many equilibrium states of the spin glass on a Bethe lattice at low temperatures is confirmed numerically, directly on the lattice. Following a method due to Nemoto and Takayama, approximate solutions of the exact equations for the Bethe lattice site magnetisations $\{m_i\}$ are found by minimising the norm $|\nabla F| \equiv [\Sigma_i (\partial F/\partial m_i)^2]^{1/2}$ where $F(\{m_i\})$ is the free energy. In order to examine the stability of solutions on the Bethe lattice using the Hessian $\partial^2 F/\partial m_i\, \partial m_j$, it is necessary to connect up boundary sites. Evidence is presented for the bifurcation of equilibrium states with decreasing temperature.

In the last decade work on the infinite-range SK model (Sherrington and Kirkpatrick 1975) of spin glasses has greatly clarified the nature of the spin-glass phase. Within the framework of the replica method (Parisi 1979a, b, 1980a, b, c) a picture has emerged of a large number of equilibrium states organised in phase space in an ultrametric structure (Mézard *et al* 1984a, b). Work directly confirming the existence of these *pure states*—by finding solutions of the self-consistent TAP equations (Thouless *et al* 1977) for the site magnetisations $\{m_i\}$ for a given bond sample $\{J_{ij}\}$—was initiated by Bray and Moore (1979) and extended more recently by Nemoto and Takayama (1985, 1986) and Nemoto (1987).

In this letter we study the *finite-range* spin glass on a Bethe lattice (Bowman and Levin 1982, Thouless 1986, Mottishaw 1987). Even though this model is still at the mean-field level, one hopes it is more realistic than the SK model (or at least provides an independent check on the mean-field behaviour). In a recent letter (Mottishaw 1987) it was shown, using the replica method, that for the spin glass on a Bethe lattice there is a replica symmetry breaking spin-glass transition. Close to the transition, the nature of the spin-glass phase, as described by the distribution $P(q)$ of overlaps between equilibrium states (see De Dominicis and Young 1983, Parisi 1983), was shown to be identical to that for the SK model.

By adopting the numerical method applied by Nemoto and Takayama to the solution of the TAP equations for the SK model, we confirm the existence of many equilibrium states directly on the Bethe lattice by finding approximate solutions of the exact equations due to Bowman and Levin (1982)—hereafter referred to as the BL equations—for the Bethe lattice site magnetisations $\{m_i\}$ for a given bond sample $\{J_{ij}\}$. Although at this preliminary stage the number of equilibrium states so generated is not large enough to explore $P(q)$ and compare it with Mottishaw's solution, or to address the question of ultrametricity, we are able to confirm that the scenario proposed by Mézard

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et al (1984a)—in which an infinite bifurcation of equilibrium states takes place as temperature decreases in the spin-glass phase—applies to the Bethe lattice.

We consider the Hamiltonian

$$H = -\sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j - \sum_i h_i\sigma_i \tag{1}$$

in which $\sigma_i = \pm 1$ are Ising spins at the sites of a Bethe lattice (figure 1), and the coupling J_{ij} between nearest-neighbour sites $\langle ij \rangle$ is chosen independently at random from the symmetric distribution $\rho(J_{ij}) = \frac{1}{2}[\delta(J_{ij} - J) + \delta(J_{ij} + J)]$. We wish to study the equilibrium states of this system, each described by a set of site magnetisations $\{m_i\}$, i.e. $m_i = \langle \sigma_i \rangle$.

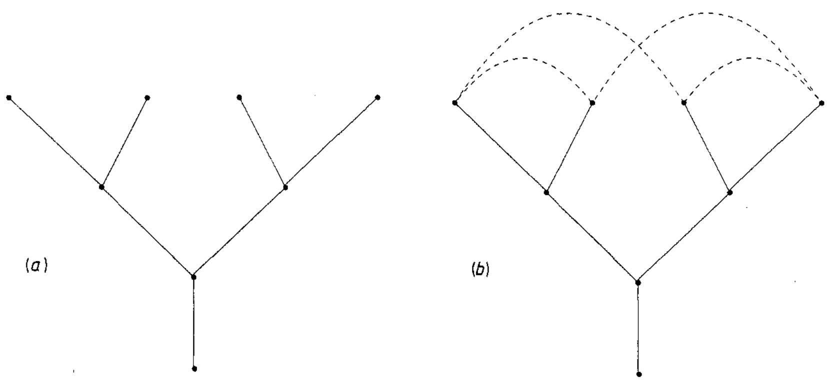

Figure 1. A subtree with branching factor $K = 2$, showing boundary sites (a) unconnected and (b) connected. In either case one may construct a tree from two such subtrees connected via a common central bond. In case (a) the result is a Cayley tree, whose central portion far from the boundary constitutes a Bethe lattice. In case (b) the result is physically equivalent to a Bethe lattice; all sites have the same number $K+1 = 3$ of nearest neighbours as in the bulk of the Cayley tree.

Bowman and Levin (1982) showed that for a given bond sample $\{J_{ij}\}$ the hierarchal tree structure of the Bethe lattice admits a set of exact equations (the BL equations) for the site magnetisations $\{m_i\}$:

$$m_i = \tanh\left[\beta h_i + \sum_j \tanh^{-1}\left(\frac{1 - g^2 - r_{ij}}{2(m_i - g_{ij}m_j)}\right)\right] \qquad i = 1, \ldots, N. \tag{2}$$

Here the sum is over the z nearest neighbours of site i, N is the number of sites in the lattice, $\beta = 1/kT$, $g_{ij} = \tanh(\beta J_{ij})$, $g^2 = g_{ij}^2$ and

$$r_{ij} = [(1 - g^2)^2 - 4g_{ij}(m_i - g_{ij}m_j)(m_j - g_{ij}m_i)]^{1/2}. \tag{3}$$

The BL equations for the spin glass on a Bethe lattice are analogues of the TAP equations for the SK model. In the limit of infinite z, the BL equations are identical to the TAP equations. Solutions of the BL equations define the spectrum of possible equilibrium states $\{m_i\}$. To calculate the free energy $F(\{m_i\})$, one may take equation (2) in the form $\partial F/\partial m_i = h_i$ and then integrate. The Hessian matrix is given by $\partial^2 F/\partial m_i \partial m_j = \partial h_i/\partial m_j$.

Before discussing solutions of the BL equations (2) we address the question of boundary sites (figure 1). Because boundary sites constitute a macroscopic fraction of the total number of sites N, the way they are connected turns out to be of crucial importance. For a Bethe lattice with branching factor K, our procedure is to (randomly) interconnect the boundary sites such that each one then has the same coordination number $z = K+1$ as that of bulk sites (figure 1(b)). In this way we naturally satisfy the physically sensible requirement that all sites are to be treated equivalently. Only then do we recover precisely the TAP equations in the limit $z \to \infty$.

More important, leaving boundary sites 'dangling' (figure 1(a)) would give a Cayley tree upon which physics is quite different. For example, for a *ferromagnetic* system Eggarter (1974) has shown that there is no phase transition on the Cayley tree, but for the central region far from the boundary there is a phase transition at $T = T_c$ (where $\tanh(J/kT_c) = 1/K$) in which a large number of sites cooperate to produce a magnetisation in a negligibly small portion of the Cayley tree. Such a region, in which all sites are equivalent, corresponds to the Bethe lattice (see, e.g., Baxter 1982). Hence for instance, in the stability analysis of the paramagnetic state on a Cayley tree, the lowest eigenvalue $\lambda_{\min}$ of the Hessian $\partial^2 F/\partial m_i \partial m_j$ remains *positive* for all $T > 0$ (which we have confirmed numerically). In the corresponding stability analysis on a Bethe lattice, however, one cannot simply calculate the Hessian in the central region of a Cayley tree because the edge of this region will artificially introduce dangling boundary sites as at the boundary of the Cayley tree itself. Hence we still obtain $\lambda_{\min} > 0$ for all $T > 0$ and so the ferromagnetic transition at T_c is not seen in this analysis (although it may be derived by other means (Baxter 1982)).

However, when $z = K+1$ on the boundary all sites are equivalent and one may easily show analytically for this system that $\lambda_{\min}$ becomes negative as T decreases through T_c; only then does the Hessian matrix faithfully reflect the physical stability of equilibrium states on the Bethe lattice. We have confirmed numerically that when $z = 1$ on the boundary the same stability analysis of the paramagnetic solution applies in the spin glass below the Almeida–Thouless (AT) line in the (h, T) plane. Since the stability analysis forms an important part of our work we have used connected boundary sites (figure 1(b)).

These considerations appear to question the validity of a previous method (Thouless 1986) of generating equilibrium states $\{m_i\}$ for the spin glass on the Bethe lattice, in which one sets up random fields on boundary sites of a Cayley tree, and then iterates shell-by-shell the exact equations equivalent to equation (2) for the effective fields $\{\xi_i\}$, defined by $m_i = \tanh(\beta\xi_i)$. One looks for fixed-point solutions $\{\xi_i\}$ well within the bulk, for which the distribution of effective fields $P(\xi)$ is invariant from shell to shell.

In this procedure, the AT line is located as the line above which $P(\xi) = \delta(\xi)$. The region below this line (the spin-glass phase) is characterised by a sensitivity of the solutions to boundary conditions and, consequently, by the existence of many fixed-point solutions for which $P(\xi)$ is non-trivial. This is valid insofar as one then correctly locates the AT line but, in the light of the above remarks, we believe it is invalid as a procedure for generating the correct stable equilibrium states below this line. What one requires are self-consistent solutions $\{m_i\}$ on the fully connected lattice with $z = K+1$ everywhere.

We now discuss solutions of the BL equations with $z = K+1$ everywhere. In zero external field, the spin-glass transition occurs at a temperature T_g given by $\tanh(J/kT_g) = 1/\sqrt{K}$. In what follows we have taken $\{h_i = 0\}$, $J = 1$ and $K = 2$ in equation (2) and have constructed a Bethe lattice from two subtrees (figure 1(b))

connected by a central bond. (Boundary sites belonging to different subtrees may be connected.) We have been guided by previous work on the TAP equations. For the TAP equations Bray and Moore (1979) found that direct iteration hardly ever converged, and we find that the same is true for the BL equations. Therefore we have adopted a more efficient procedure first used by Nemoto and Takayama (1985) to generate approximate solutions to the TAP equations.

This involves minimising the norm $|\nabla F| \equiv [\Sigma_i (\partial F/\partial m_i)^2]^{1/2}$ with respect to $\{m_i\}$, where $F(\{m_i\})$ is the free energy defined by integrating the BL equations. A standard NAG routine is used for the minimisation. As discussed by Nemoto and Takayama, there are two possibilities for the outcome of this procedure, according to whether it stops with (i) $\min|\nabla F| = 0$ or (ii) $\min|\nabla F| > 0$. In case (i), the configuration $\{m_i\}$ is an exact solution of the BL equations. In case (ii), the configuration $\{m_i\}$ represents a turning point of F, i.e. the lowest eigenvalue $\lambda_{\min}$ of the Hessian matrix $\partial^2 F/\partial m_i \partial m_j$ is zero identically, but is not an exact solution of the BL equations.

Following Nemoto and Takayama, in case (i) we have examined the N dependence of the lowest eigenvalue $\lambda_{\min}$ of the Hessian (figure 2(a)). In case (ii) we have examined the N dependence of $\min|\nabla F|$ (figure 2(b)). These numerical results are consistent with the limiting result that, as $N \to \infty$, $\lambda_{\min} \to 0$ in case (i) and $\min|\nabla F| \to 0$ in case (ii), although this evidence is by no means conclusive. (Similar results for the SK model were reported by Nemoto and Takayama (1985).) In combination, these results imply that in the thermodynamic limit the minimising configurations $\{m_i\}$ represent equilibrium states (solutions of the BL equations) which are marginally stable ($\lambda_{\min} = 0$).

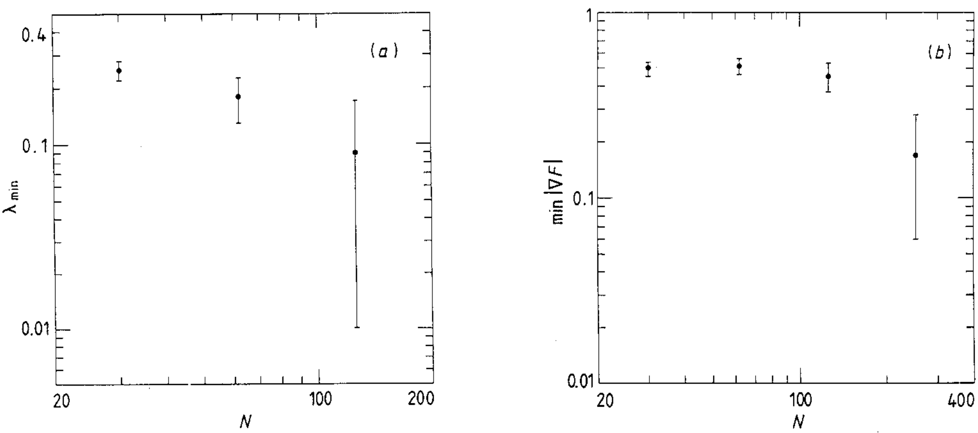

Figure 2. (a) $\lambda_{\min}$ plotted against N for exact solutions of the BL equations. Data are averaged over 20, 12 and 8 samples on lattices with $N = 30$, 62 and 126 respectively. (b) $\min|\nabla F|$ plotted against N for approximate solutions of the BL equations. Data are averaged over 100, 50, 30 and 15 samples on lattices with $N = 30$, 62, 126 and 254 respectively.

With this procedure we have constructed a picture consistent with the existence of many equilibrium states below T_g. At $T = 0.4T_g$, for a particular bond sample on a lattice with $N = 62$ we generated 100 configurations which minimised $|\nabla F|$, from which 20 configurations were chosen with significant weight, i.e. they satisfied the condition $w_i \equiv \exp(-\beta F_i)/\Sigma_j \exp(-\beta F_j) > 10^{-5}$, where F_i is the free energy of the ith configuration. We then raised the temperature in steps of $0.02T_g$ and at each temperature new equilibrium states were generated by using the equilibrium states at the previous temperature as inputs for the minimisation of $|\nabla F|$.

Figure 3 shows schematically how the equilibrium states evolve through phase space as the temperature is raised. A state evolves continuously in phase space with increasing temperature until at some particular temperature it jumps discontinuously, where a confluence of states may occur. All states eventually evolve to the paramagnetic state. Conversely, as temperature decreases, one may interpret this evolution as evidence of an infinite bifurcation of equilibrium states (Mézard *et al* 1984a, b). In arriving at these conclusions, we have been guided by the paper of Nemoto and Takayama (1986) in which similar results were reported for the SK model.

In conclusion, using the method of Nemoto and Takayama to generate approximate solutions $\{m_i\}$ of the BL equations, we have presented numerical evidence for the existence of many equilibrium states of the spin glass on a Bethe lattice at low temperature. These states are marginally stable in the thermodynamic limit and undergo a sequence of phase transitions with decreasing temperature.

In addition we have shown that, in order to examine the stability of solutions of the BL equations on a Bethe lattice, boundary sites must be interconnected. Note that our results for the fully connected lattice are consistent with the result (Mottishaw 1987) that replica symmetry is broken in the spin-glass phase only if the effective fields on boundary sites are correlated. This latter result has also been confirmed by the recent Monte Carlo work of Lai and Goldschmidt (1989).

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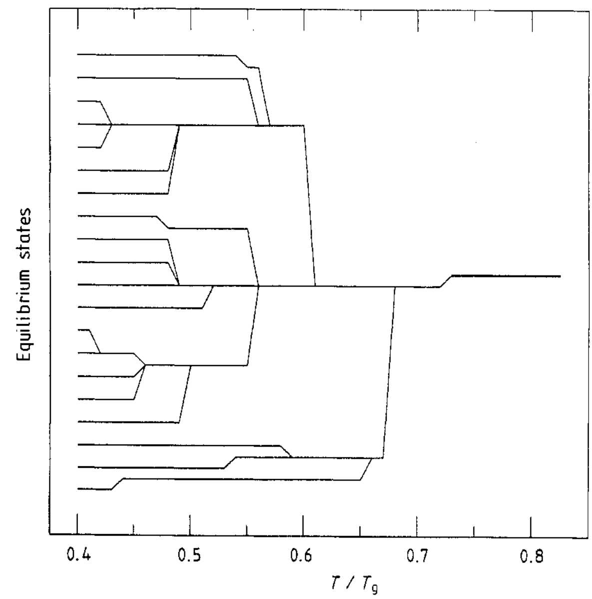

Figure 3. Schematic diagram illustrating the phase-space evolution of equilibrium states with temperature. A horizontal line represents the continuous evolution of a particular state, a sloping line represents its discontinuous evolution, and the bold line represents the paramagnetic state. Details are given in the text.

References

Baxter R J 1982 *Exactly Solved Models of Statistical Mechanics* (Academic: New York)
Bowman D and Levin K 1982 *Phys. Rev.* B **25** 3438
Bray A J and Moore M A 1979 *J. Phys. C: Solid State Phys.* **12** L441
De Dominicis C and Young A P 1983 *J. Phys. A: Math. Gen.* **16** 2063
Eggarter T P 1974 *Phys. Rev.* B **9** 2989
Lai P-Y and Goldschmidt Y Y 1989 *J. Phys. A: Math. Gen.* **22** to be published
Mézard M, Parisi G, Sourlas N, Toulouse G and Virasoro M 1984a *Phys. Rev. Lett.* **52** 1156
—— 1984b *J. Physique* **45** 843
Mottishaw P 1987 *Europhys. Lett.* **4** 333
Nemoto K 1987 *J. Phys. C: Solid State Phys.* **20** 1325
Nemoto K and Takayama H 1985 *J. Phys. C: Solid State Phys.* **18** L529
—— 1986 *J. Magn. Magnet. Mater.* **54–57** 135
Parisi G 1979a *Phys. Lett.* **73A** 203
—— 1979b *Phys. Rev. Lett.* **43** 1754
—— 1980a *J. Phys. A: Math. Gen.* **13** L115
—— 1980b *J. Phys. A: Math. Gen.* **13** 1101
—— 1980c *J. Phys. A: Math. Gen.* **13** 1807
—— 1983 *Phys. Rev. Lett.* **50** 1946
Thouless D J 1986 *Phys. Rev. Lett.* **56** 1082
Thouless D J, Anderson P W and Palmer R G 1977 *Phil. Mag.* **35** 593
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792